INTRODUCTION TO FREQUENCY RESPONSE

Frequency response is a valuable tool in the analysis and design of control systems.

Substitution Rule:

Consider a simple first-order system with transfer function,

$$G(s) = \frac{1}{\tau s + 1} \quad \text{———} \quad (1)$$

Substitute, $s = j\omega$

$$G(j\omega) = \frac{1}{j\omega\tau + 1}$$

Converting this expression to polar form by multiplying numerator and denominator by the conjugate of $(j\omega\tau + 1)$.

$$G(j\omega) = \frac{(-j\omega\tau + 1)}{(j\omega\tau + 1)(-j\omega\tau + 1)} = \frac{1}{1 + \omega^2\tau^2} - j\frac{\omega\tau}{1 + \omega^2\tau^2} \quad \text{———} \quad (2)$$

To convert a complex no. in rectangular form $(z = a + jb)$ to polar form $(|z| \measuredangle z)$ one uses the relationships:

$$|z| = \sqrt{a^2 + b^2} \quad \text{and} \quad \measuredangle z = \tan^{-1}\frac{b}{a}$$

Applying these relationships to eq (2) gives,

$$G(j\omega) = \frac{1}{\sqrt{\omega^2\tau^2 + 1}} \measuredangle \tan^{-1}(-\omega\tau) \quad \text{———} \quad (3)$$

The response of a first-order system to a sinusoidal input of frequency ω is also a sinusoid of frequency ω. The ratio of amplitude of the response to that of the input is $1/\sqrt{\omega^2\tau^2 + 1}$

The phase difference between output and input is $\tan^{-1}(-\omega\tau)$.

$$AR = |G(j\omega)|$$

$$\text{phase angle} = \measuredangle G(j\omega)$$

i.e. To obtain the amplitude ratio (AR) and phase angle, $j\omega$ is substituted for s in the transfer function and then takes the magnitude and argument (or angle) of the resulting complex number respectively.

Example 1

1. The pertinent transfer function is $G(s) = \frac{1}{0.1s+1}$

The frequency of the bath-temp. variation is given as $10/\pi \frac{\text{cycles}}{\text{min}}$ which is equivalent to 20 rad/min.

Solution - $s = j\omega$, $\omega = 20$ rad/min

$S = 20j$

$$G(20j) = \frac{1}{(0.1)(20j)+1} = \frac{1}{2j+1}$$

In polar form,

$$G(20j) = \frac{1}{\sqrt{2^2+1^2}} = \frac{1}{\sqrt{5}}, \quad \measuredangle G(20j) = \tan^{-1}(-\omega T)$$

$$= \tan^{-1}(-20T)$$

$$= -63.5^\circ$$

Generalization

For n^{th} order linear system

$$a^n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots\cdots + a_1 \frac{dy}{dt} + a_0 y = X(t) \longrightarrow \text{(I)}$$

$y \rightarrow$ o/p variable , $X(t)$ – forcing function or input variable

If $X(t)$ is sinusoidal,

$$X(t) = A \sin \omega t$$

$$y_p(t) = C_1 \sin \omega t + C_2 \cos \omega t$$

If the system is stable, the roots of above characteristic eqn (I) all lie to the left of the imaginary axis and the complementry solution will vanish exponentially in time. Then y_p is the quantity previously defined as the sinusoidal or frequency response. If the system is not stable, the complementary solution grows exponentially and the term frequency response has no physical significance. because $y_p(t)$ is inconsequential.

The evaluation of C_1 and C_2. The amplitude and phase of $Y_p(t)$.

$$Y_p = B_1 \sin(\omega t + B_2)$$

To change $X(t)$ and $Y_p(t)$ from trigonometric to exponential form, using the identity.

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Thus,

$$X(t) = \frac{A}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

$$Y_p(t) = \frac{B_1}{2j}\left[e^{j(\omega t + B_2)} - e^{-j(\omega t + B_2)}\right]$$

$$\frac{B_1 e^{j(\omega t + B_2)}}{2j}\left[a_n (j\omega)^n + a_{n-1}(j\omega)^{n-1} + \dots + a_1 (j\omega) + a_0\right]$$

$$- \frac{B_1 e^{-j(\omega t + B_2)}}{2j}\left[a_n(-j\omega)^n + a_{n-1}(-j\omega)^{n-1} + \dots + a_1(-j\omega) + a_0\right] = \frac{A}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$

The coefficients of $e^{j\omega t}$ on both sides of above eq. must be equal.

Hence,

$$B_1 e^{jB_2}\left[a_n (j\omega)^n + a_{n-1}(j\omega)^{n-1} + \dots + a_1 (j\omega) + a_0\right] = A$$

Above eqn. is satisfied if

$$\left|\frac{1}{a_n (j\omega) + a_{n-1}(j\omega)^{n-1} + \dots + a_1 (j\omega) + a_0}\right| = \frac{B_1}{A}$$

$$\angle \frac{1}{a_n (j\omega)^n + a_{n-1}(j\omega)^{n-1} + \dots + a_1 (j\omega) + a_0} = B_2 = \text{Phase angle}$$

But $\frac{B_1}{A}$ and B_2 are the AR and phase angle of the response respectively. The transfer function relating X and Y is,

$$\frac{Y(s)}{X(s)} = \frac{1}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

Example 2: Find the frequency response of the system with the general second order transfer function.

$$\frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

Putting $s = j\omega$ yields

$$\frac{1}{1 - \tau^2\omega^2 + 2j\zeta\omega\tau}$$

In polar form

$$AR = \frac{1}{\sqrt{(1-\omega^2\tau^2)^2 + (2\zeta\omega\tau)^2}} \angle \tan^{-1}\left(\frac{-2\zeta\omega\tau}{1-\omega^2\tau^2}\right)$$

Hence,

$$AR = \frac{1}{\sqrt{(1-\omega^2\tau^2)^2 + (2\zeta\omega\tau)^2}}$$

$$\text{phase angle} = \tan^{-1}\left(\frac{-2\zeta\omega\tau}{1-\omega^2\tau^2}\right)$$

Transportation Lag:

Transportation lag is described by the relation,

$$Y(t) = X(t - \tau)$$

The output Y lags the input X by an interval of time τ.
If X is sinusoidal,

$$X = A \sin \omega t$$

then, $Y = A \sin \omega (t - \tau) = A \sin (\omega t - \omega\tau)$

The transfer function is,

$$G(s) = \frac{Y(s)}{X(s)} = e^{-\tau s}$$

$$s = j\omega, \quad G(j\omega) = e^{-j\omega\tau}$$

$$AR = \left| e^{-j\omega\tau} \right| = 1$$

$$\text{Phase angle} = \angle e^{-j\omega\tau} = -\omega\tau$$

$\therefore$ the validity of the rule is verified.

Example 3 : The stirred - tank heater has a capacity of 15 gal. water is entering and leaving the tank at the constant rate of 600 lb/min. The heated water that leaves the tank enters a well - insulated section of 6 - in - ID pipe. Two feet from the tank, a thermocouple is placed in this line for recording the tank temp. as in fig. The electrical heat input is held constant at 1,000 kw. If the inlet temp. is varied according to the relation.

$$T_i = 75 + 5 \sin 46t$$

where, T_i - is in °F and t is in minutes. Find the eventual behaviour of the thermocouple reading T_m. Compare this with the behaviour of the tank temp. T. (Assume that the thermocouple has a very small time constant and effectively measures the true fluid temp. at all times).

Solution

It is required to find the frequency response of T_m to T_i.

Deviation variable, $T_i' = T_i - 75 = 5 \sin 46t$.

To define a deviation variable for T_m, if T_i were held at 75 °F T_m would come to the s.s. satisfying.

$$q_s = wc (T_{ms} - T_{is})$$

$$\Rightarrow \frac{q_s}{wc} = T_{ms} - T_{is}$$

$$T_{ms} = \frac{q_s}{wc} + T_{is} = \frac{(1000 \times 1000)(0.0569)}{(600)(1.0)} + 75$$

$$= 170 \,°F$$

$$T_m' = T_m - 170$$

The overall system between T_i' and T_m' is made up of two components in series: the tank and the 2-ft section of pipe. The transfer function for the tank is,

$$G_1(s) = \frac{1}{\tau_1 s + 1}$$

WKT, $\tau_1 = \frac{\rho V}{w} = \frac{(60.3)(15)}{(600)(7.48)} = 0.202$ min

The transfer function of 2-ft section of pipe, which corresponds to a transportation lag is,

$$G_2(s) = e^{-\tau_2 s}$$

where, τ_2 is the length of time required for the fluid to transverse the length of pipe.

$$\tau_2 = \frac{L}{V} = \frac{(2)(60.3)(0.197)}{600} = 0.0396 \text{ min.}$$

$$A_p = \frac{\pi d^2}{4} = 0.197$$

Since the two systems are in series, the overall Transfer function between T_i' and T_m' is,

$$\frac{T_m'}{T_i'} = \frac{e^{-\tau_2 s}}{\tau_1 s+1} = \frac{e^{-0.0396s}}{0.202s+1}$$

The overall transfer function is the product of the individual T.F, hence its magnitude will be the product of the magnitudes and its argument the sum of the arguments of the individual T.F.

In general,

$$G(s) = G_1(s)\, G_2(s) \ldots\ldots\ldots\, G_n(s)$$

then, $|G(j\omega)| = |G_1(j\omega)|\,|G_2(j\omega)| \ldots\ldots |G_n(j\omega)|$

$$\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega) + \ldots\ldots + \angle G_n(j\omega)$$

(Assume $s = 46j$).

For tank,

$$AR = \frac{1}{\sqrt{(46)^2 \times (0.202)^2 + 1}} = \frac{1}{9.35} = 0.107$$

phase angle $= \tan^{-1}(-46)(0.202) = -84°$

For pipe: $AR = 1$

Phase angle $= -\omega\tau_2 = -(46)(0.0396) = -1.82$ rad

$= -104°$.

Overall AR = (1)(0.107) = 0.107.

Overall phase lag from T_i' to T_m' is sum of the individual lags,

$$\measuredangle \frac{T_m'}{T_i'} = -84 - 104 = -188^\circ$$

Hence,

$$T_m = 170 + 0.535 \sin(46t - 188^\circ)$$

where, $T' =$ tank temp $- 170^\circ F$

For comparision, a plot of T_i', T_m' and T' is given in fig. where, $T =$ tank temp $- 170^\circ F$

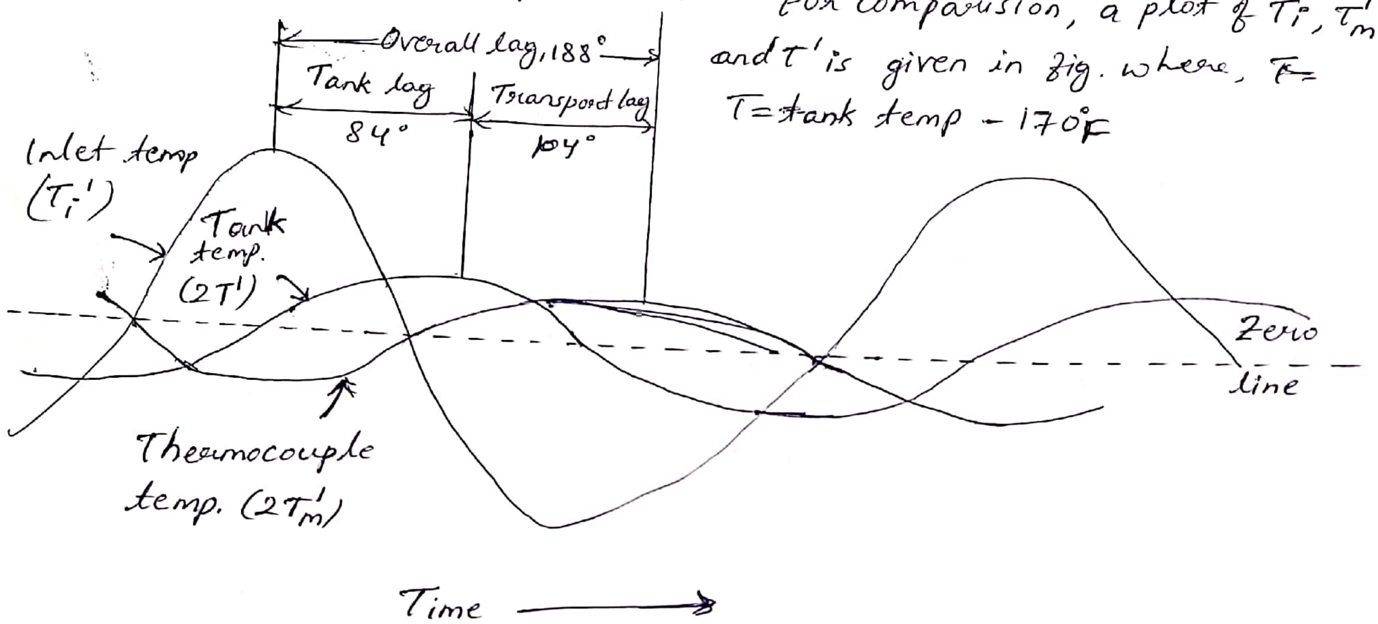
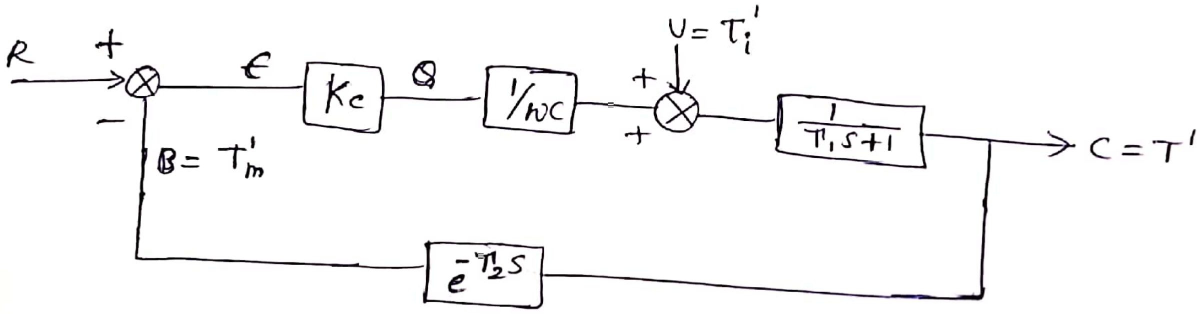


Fig - Temp. variation.



In the block diagram, T_i' is replaced by U, T' by C. T_m' by B to conform with standard block diagram nomenclature. The heat being added to the tank is given in deviation variables as $-K_c B$. For above fig, which shows that response of the uncontrolled tank to a sinusoidal variation in U, it can be seen that the peaks of U and B are almost exactly opposite because the phase difference is 188°.

If the loop were closed, the control system would have a tendency to add more heat when the inlet temp T_i is at its high peak, because B is then negative and $-K_c B$ becomes positive. ($\because R \rightarrow$ constant at zero).

Conversely, when the inlet temp. is at a low point, the tendency will be for the control system to add less heat because B is positive. This is precisely opposite to the way the heat input should be controlled.

Therefore, the possibility of an unstable control system exists for this particular sinusoidal variation in frequency. Indeed, if K_c is taken too large, the tank temp. will oscillate with increasing amplitude for all variations in U and hence an unstable control system exists.

BODE DIAGRAMS

W.K.T. AR & Phase lag are functions of frequency. There is a convenient graphical representation of their dependence on the frequency that largely eliminates direct calculation. This is called a Bode diagram and consists of two graphs: logarithm of AR versus logarithm of frequency, and phase angle versus logarithm of frequency.

First-order system:

The AR and phase angle for the sinusoidal response of a first-order system are.

$$AR = \frac{1}{\sqrt{\tau^2\omega^2 + 1}} \rightarrow ①$$

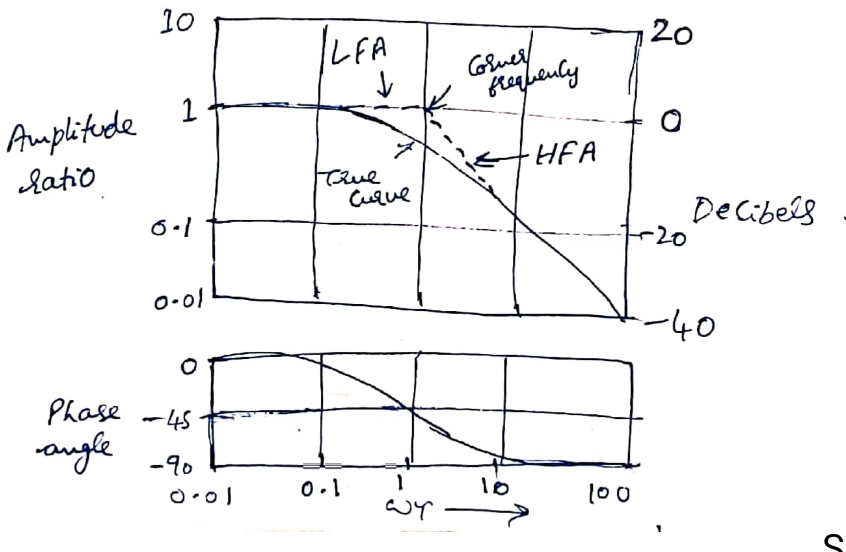
$$\text{Phase angle} = \tan^{-1}(-\omega\tau) \rightarrow ②$$

These functions are taken as $\omega\tau$ for the purpose of generality.

$$\log AR = -\frac{1}{2}\log[(\omega\tau)^2 + 1)] \rightarrow ③$$

The first part of the Bode diagram is a plot of equ. ③. The true curve is shown as the solid line on the upper part of fig. Some asymptotic considerations can simplify this plot, As $(\omega\tau) \rightarrow 0$, eqn. ① shows that $AR \rightarrow 1$. This is indicated by the low-frequency asymptote on fig. As $(\omega\tau) \rightarrow \infty$, equ. ③ becomes asymptotic to

$$\log AR = -\log(\omega\tau)$$



which is a line of slope -1, passing thro' the pt. $\omega\tau = 1$, $AR = 1$

This line is indicated as the high-frequency asymptote in fig.

The frequency $\omega_c = \frac{1}{\tau}$, where the two asymptotes intersect, is known as the "corner frequency". The deviation of the true AR curve from the asymptotes is a maximum at the corner frequency.

using $\omega_c = \frac{1}{\tau}$ in eqn. ① gives.

$$AR = \frac{1}{\sqrt{2}} = 0.707 \quad \text{as the true value,}$$

whereas the intersection of the asymptotes occurs at $AR = 1$.

In the lower half of fig, the phase curve is given by eqn. ②. since,

$$\phi = \tan^{-1}(-\omega\tau) = -\tan^{-1}(\omega\tau)$$

It is evident that ϕ approaches $0°$ at low frequencies and $-90°$ at high frequencies. This verifies the low and high frequency portions of the phase curve. At the corner frequency, $\omega_c = \frac{1}{\tau}$,

$$\phi_c = -\tan^{-1}(\omega_c\tau) = -\tan^{-1}(1) = -45°$$

There are asymptotic approximations available for the phase curve, but they are not so accurate or so widely used as those for the AR. Instead, it is convenient to note that the curve is symmetric about $-45°$.

AR (or gains) are reported in decibels.

$$\underset{(db)}{\text{Decibels}} = 20 \log_{10}(AR)$$

Thus, an AR of unity corresponds to zero decibels and an amplitude ratio of 0.1 corresponds to -20 decibels.

First-order systems in series.

Frequency response of systems in series.

$$AR = \frac{1}{\sqrt{\omega^2\tau_1^2+1}\sqrt{\omega^2\tau_2^2+1}} \longrightarrow (A)$$

$$\phi = \tan^{-1}(-\omega\tau_1) + \tan^{-1}(-\omega\tau_2) \longrightarrow (B)$$

Since the AR is plotted on a logarithmic basis, multiplication of the ARs is accomplished by addition of logarithms on the Bode diagram. The phase angles are added directly.

1) Plot the Bode diagram for the system whose overall transfer function is.

$$\frac{1}{(s+1)(s+5)}$$

$$\frac{\frac{1}{5}}{(s+1)\left(\frac{1}{5}s+1\right)}$$

$\tau_1 = 1$, $\tau_2 = \frac{1}{5}$.

$$AR = \frac{1/5}{\sqrt{\omega^2+1}\ \sqrt{(\omega/5)^2+1}}.$$

$$\log AR = \log\frac{1}{5} - \frac{1}{2}\log(\omega^2+1) - \frac{1}{2}\log\left[\left(\frac{\omega}{5}\right)^2+1\right]$$

or

$$\log AR = \log\frac{1}{5} + \log AR_1 + \log AR_2.$$

where $(AR)_1$, $(AR)_2$ are the ARs of the individual first-order systems, each with unity gain.

Above eqn. shows that the overall AR is obtained, on logarithmic coordinates, by adding the individual ARs and a constant corresponding to the steady-state gain.

The individual ARs must be plotted as functions of $\log\omega$ rather than $\log(\omega\tau)$ because of the different time constants. This is easily done by shifting the curves of fig to the right & left so that the corner frequency falls at $\omega = 1/\tau$.

Graphical Rules for Bode Diagrams.

Consider a no. of systems in series. The overall AR is the product of the individual AR'S, and the overall phase angle is the sum of the individual phase angles.

$$\therefore \log AR = \log(AR)_1 + \log(AR)_2 + \dots + \log(AR)_n$$

&

$$\phi = \phi_1 + \phi_2 + \dots + \phi_n$$

where, $n \to$ total no. of systems. Therefore, the following rules apply to the true curves & to the asymptotes on the Bode diagram.

1. The overall AR is obtained by adding the individual ARS. For this graphical addition, an individual AR that is above unity on the frequency response diagram is taken as positive; an AR that is below unity is taken as negative. The logarithm of a no. greater than one is true and the logarithm of a no. less than one is negative. A convenient way to combine two or more individual AR curves is to use a pair of dividers to transfer distances at a selected value of ω.

2. The overall phase angle is obtained by addition of the individual phase angles.

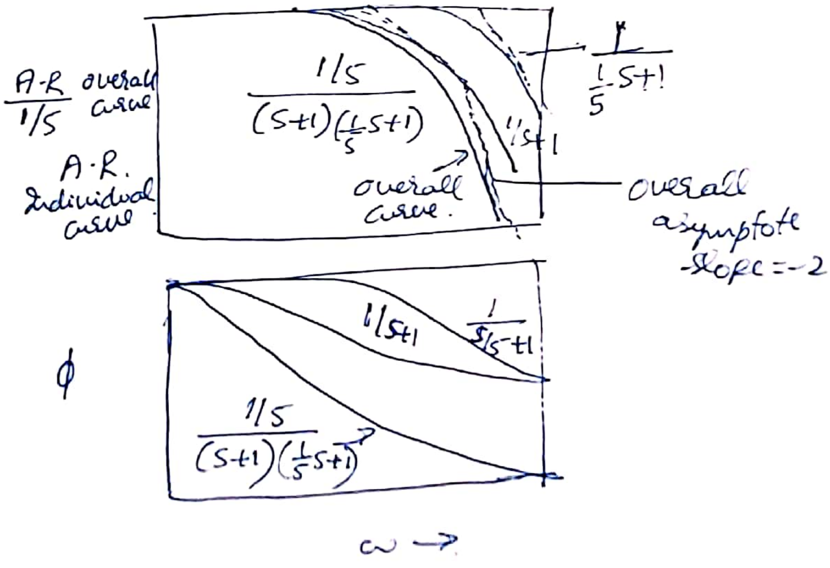
3. The presence of a constant in the overall transfer function shifts the entire AR curve vertically by a constant amount and has no effect on the phase angle. It is usually more convenient to include a constant factor in the definition of the ordinate.

Thus, the individual curves of next fig. are placed so that the corner frequencies fall at $\omega c_1 = 1$ & $\omega c_2 = 5$. These curves are added to obtain the overall curve shown. In this case the logarithms are negative and the addition is downward.

To complete the AR curve, the factor $\log \frac{1}{5}$ should be added to the overall curve. This would have the effect of shifting the entire curve down by a constant. Instead of doing this, the factor $\frac{1}{5}$ is incorporated by plotting the overall curve as AR/ 1/5 instead of AR.

Asymptotes are indicated in fig. The sum of the individual asymptotes gives the overall asymptote, which is seen to be a good approximation to the overall curve. The overall asymptote has a slope of zero below $\omega = 1$, -1 for ω b/w 1 & 5, & -2 above $\omega = 5$. Its slope is obtained by simply adding the slopes of the individual asymptotes.

To obtain the phase angle, the individual phase angles are plotted & added according to eqn. (B). The factor $\frac{1}{5}$ has no effect on the phase angle, which approaches $-180°$ at high frequency.



The second-order system.

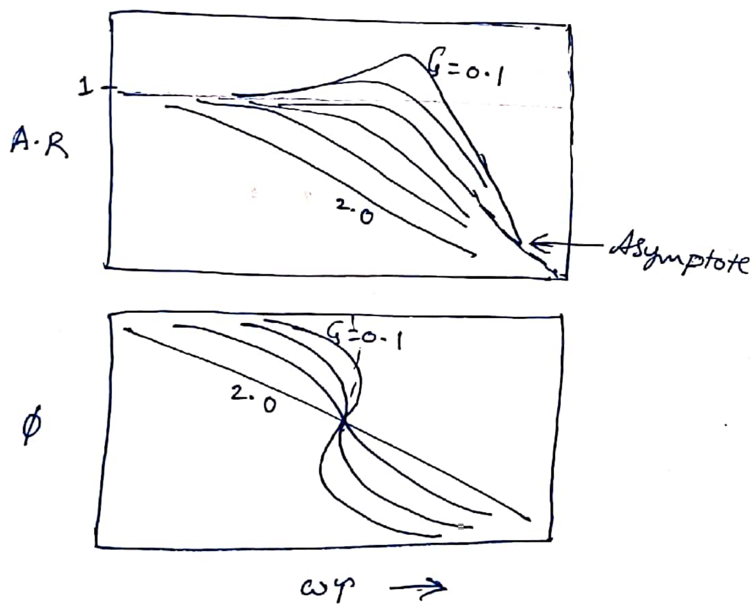
The frequency response of a system with a second-order transfer function.

$$G(s) = \frac{1}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

is given as.

$$AR = \frac{1}{\sqrt{(1-\omega^2\tau^2)^2 + (2\zeta\omega\tau)^2}} \rightarrow (a)$$

$$\text{Phase angle} = \tan^{-1}\frac{-2\zeta\omega\tau}{1-(\omega\tau)^2} \rightarrow (c')$$



For $\omega\tau << 1$, the AR or gain approaches unity.
$\omega\tau >> 1$, the AR becomes asymptotic to the line

$$AR = \frac{1}{(\omega\tau)^2}$$

This asymptote has slope -2 & intersects the line $AR = 1$ at $\omega\tau = 1$.

For $\zeta \geq 1$, second-order system is equivalent to two first-order systems in series. The fact that the AR for $\zeta \geq 1$ (as well as for $\zeta < 1$) attains a slope of -2 and phase of -180 is, therefore consistent.

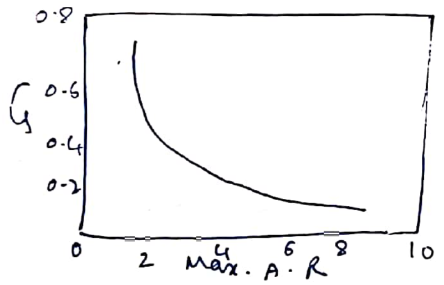
Fig shows that for $\zeta < 0.707$, the AR curves attain maxima in the vicinity of $\omega\tau = 1$. This can be checked by differentiating the expression for the AR w.r.t. $\omega\tau$ and setting the derivative to zero.

$$(\omega\tau)_{max} = \sqrt{1 - 2\zeta^2} \qquad \zeta < 0.707. \rightarrow (D)$$

for the value of $\omega\tau$ at which the max. AR occurs.

The value of the max. AR, obtained by substituting $(\omega\tau)_{max}$ into equ. (C) is

$$(AR)_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad \zeta < 0.707$$



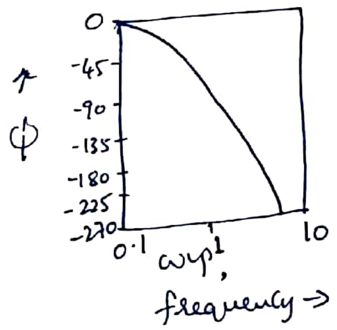
The frequency at which the max. AR is attained is called the resonant frequency & obtained from equ. (D)

$$\omega_r = \frac{1}{\tau}\sqrt{1-2\zeta^2}$$

Transportation Lag

The frequency response for

$$G(s) = e^{-\tau s} \text{ is}$$



$AR = 1$

$\phi = -\omega\tau$ radians, or $\phi = -57.2958\,\omega\tau$ degrees.

AR plot is not reqd. as it is const. at 1.0.

The transportation lag contributes a phase lag, which increases without bound as ω increases.

It is necessary to convert $\omega\tau$ from radians to degree to prepare above plot.

Proportional - Derivative Controller

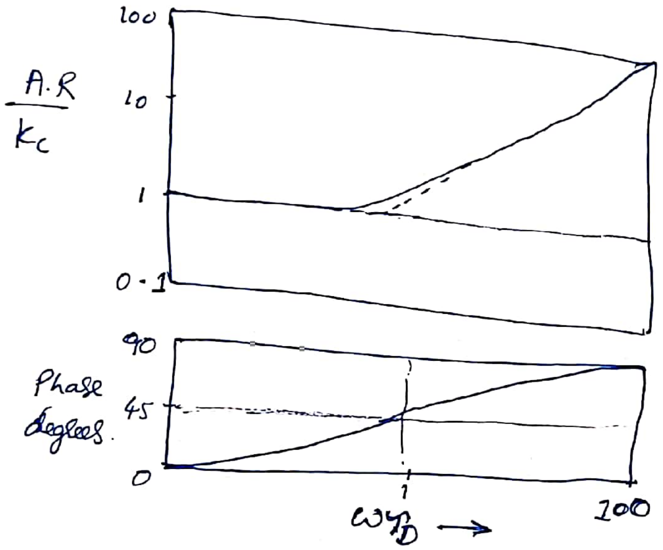
The T.F is

$$G(s) = K_c(1 + \tau_D s)$$

This has amplitude and phase behavior that is just the inverse of the first order system.

$$\frac{1}{\tau s+1}$$

The corner frequency is $\omega_c = \frac{1}{\tau_D}$



This system is important because it introduces phase lead. Thus, it can be seen that using PD control for same (Tank temp.) control systems would decrease the phase lag at all frequencies. In particular, 180° of phase lag would not occur until a higher frequency. This may exert a stabilizing influence on the control system.

Proportional Controller

A Proportional Controller with transfer function K_c has at all amplitude ratio K_c and phase angle zero for this frequencies. No Bode diagram is necessary component.

Proportional- Integral Controller

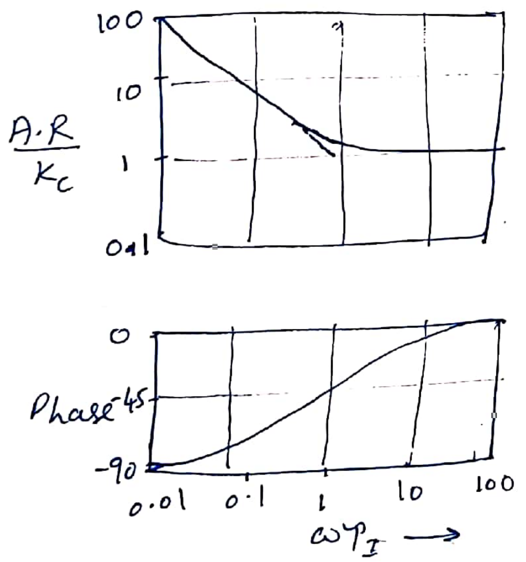
This component has the ideal transfer function

$$G(s) = K_c\left(1+\frac{1}{\tau_I s}\right)$$

$$AR = |G(j\omega)| = K_c\left|1+\frac{1}{\tau_I j\omega}\right| = K_c\sqrt{1+\frac{1}{(\omega\tau_I)^2}}$$

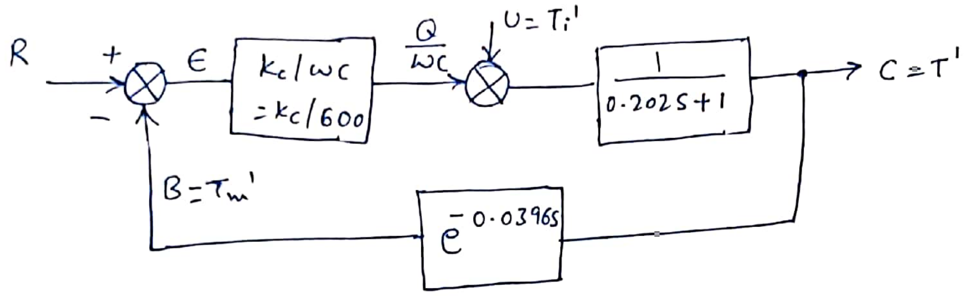
$$\text{Phase} = \measuredangle G(j\omega) = \measuredangle\left(1+\frac{1}{\tau_I j\omega}\right) = \tan^{-1}\left(-\frac{1}{\omega\tau_I}\right)$$

Corner frequency, $\omega_c = \frac{1}{\tau_I}$.



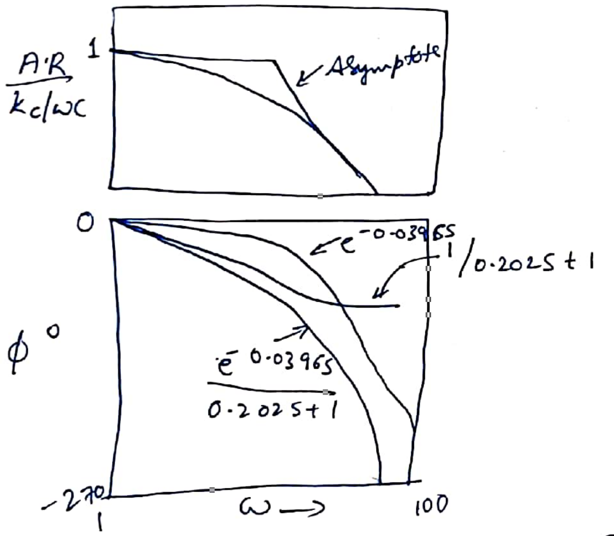
Control System Design by Frequency Response

Tank - Temp. Control System.



O. L. T. F.

$$G(s) = \frac{(k_c/600)\,e^{-0.0396s}}{0.202s+1}$$



$$\frac{AR}{k_c/600} = 0.12$$

If a proportional gain of 5000 Btu/hr °F is used,

$$AR = 0.12\,\frac{(5000)}{600}$$

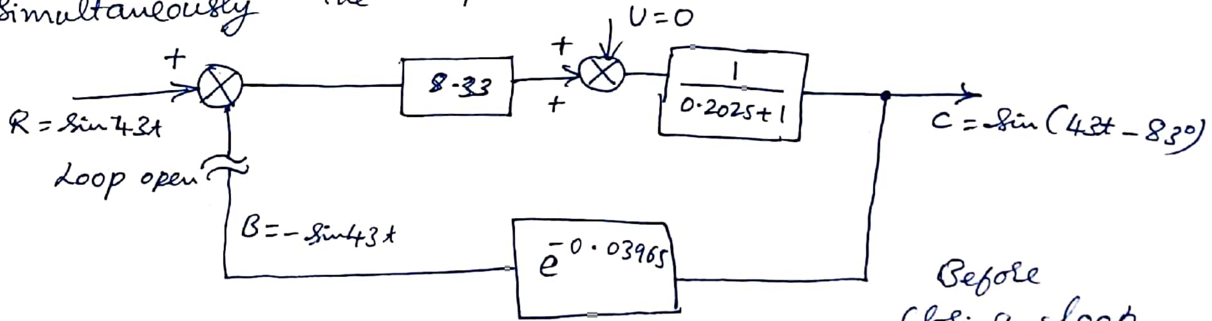
This is the AR b/w the signals ε & B.

Imagine a set point disturbance, $R = \sin 43t$. is applied to the opened loop.

Then, since the open-loop AR and phase lag are unity and $180°$.

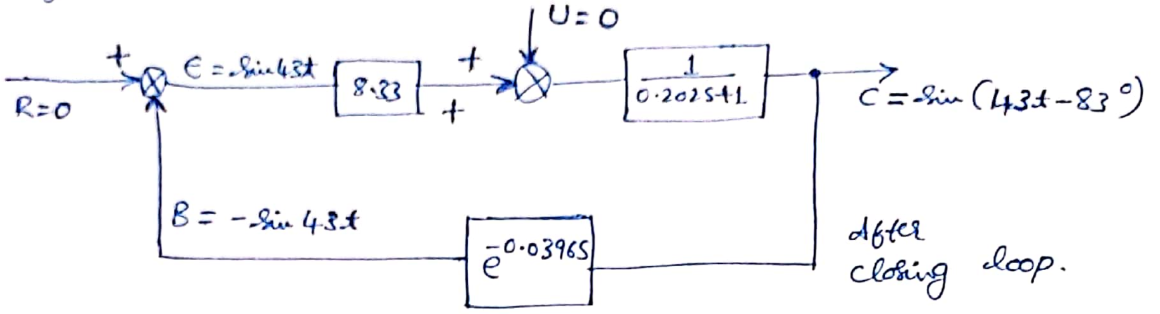
$$B = \sin(43t - 180°) = -\sin 43t$$

At same instant of time, R is set to zero. & simultaneously the loop is closed.



Before Closing loop

Fig. indicates that the closed loop Continues to oscillate indefinitely. This oscillation is theoretically sustained even though both R & U are zero.



By changing K_c (eg. $K_c = 5001$), the phase angle relations are not affected. ~~By~~ ~~Cla~~

For $K_c > 5000$, the response is unbounded, since it oscillates with increasing amplitude.

The Control System is unstable for $K_c > 5000$ because it exhibits an unbounded response to the bounded input. (The bounded input is zero in this case. for $U = R = 0$).

the condition $K_c > 5000$ corresponds to

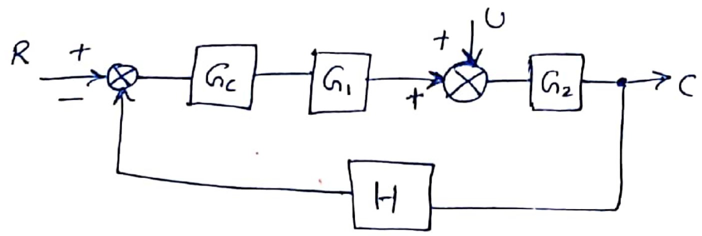
$$AR > 1$$

for the open-loop transfer function, at the frequency 43 rad/min, where the open loop phase lag is 180°.

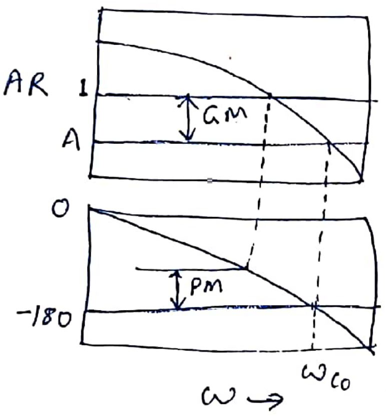
The Bode Stability Criterion

"A Control system is unstable if the open-loop frequency response exhibits an AR exceeding unity at the frequency for which the phase lag is 180°. The this frequency is called the crossover frequency. This rule is called the Bode Stability Criterion.

Gain and Phase Margins



Consider the general problem of selecting $G_c(s)$ for the system. Suppose the open-loop frequency response, when a particular controller $G_c(s)$ is tried as shown in the Bode diagram of fig below.



GM → Gain margin

PM → Phase margin

The crossover frequency, at which the phase lag is 180°, is noted as ω_{co} on the Bode diagram. At this frequency, the AR is A. If A exceeds unity, from the Bode criterion, the system is unstable and a poor selection of $G_c(s)$ has been made. In above fig. it is assumed that A is less than unity and therefore the system is stable.

If 'A' is only slightly less than unity the system is 'almost unstable' and may be expected to behave in a highly oscillatory manner even though it is theoretically stable. The constant A is determined by physical parameters of the system, such as time constants. These can be only estimated and may actually change slowly with time because of wear or corrosion. Hence, a design for which A is close to unity does not have an adequate safety factor.

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To quantitatively measure these considerations, the concept of gain margin is introduced.

$$\text{Gain margin} = \frac{1}{A}$$

Typical specifications for design are that the gain margin should be greater than 1.7. This means that the AR at crossover could increase by a factor of 1.7 over the design value before the system became unstable. The design value of the gain margin is really a safety factor.

As such, its value varies considerably with the application and designer. A gain margin of unity or less indicates an unstable system.

Another margin frequently used for design is the phase margin. As in fig, it is the difference b/w 180° and the phase lag at the frequency for which the gain is unity.

The phase margin therefore represents the additional amount of phase lag required to destabilize the system, just as the gain margin represents the additional gain for destabilization. Typical design specifications are that the phase margin must be greater than 30°. A negative phase margin indicates an unstable system.

Eg : 17-1,

17-2

Controller Tuning:

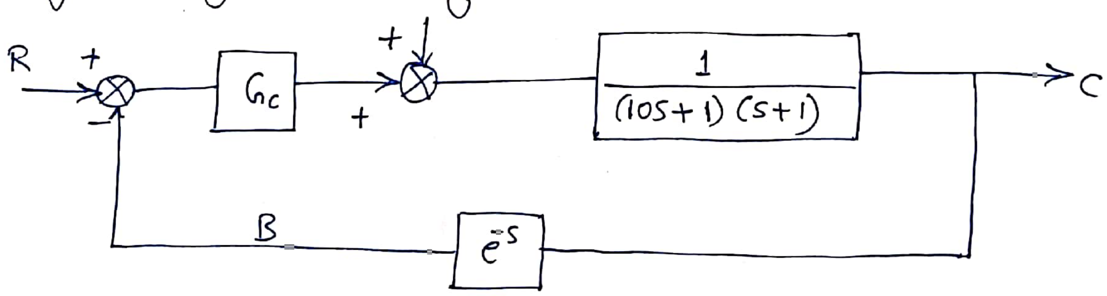
The selection of a controller type (P, PI, PID) and its parameters (K_c, τ_I, τ_D) is intimately related to the model of the process to be controlled. The adjustment of the controller parameters to achieve satisfactory control is called tuning.

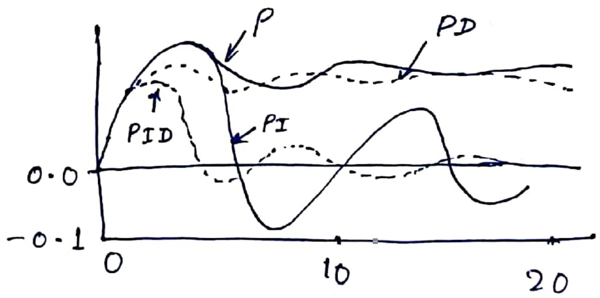
A typical criterion for good control is that the response of the system to a step change in set pt. or load should have minimum overshoot and one-quarter decay ratio. Also minimum rise time & minimum settling time.

Determining the model of a process experimentally is referred to as process identification.

Selection of Controller modes:

Consider a typical loop. Process → second order. Measuring element is transport lag, T.F of value is 1. Load response for this process for 4 types of controllers (P, PD, PI, PID). For each response curve, the process was subjected to a unit-step change in load. ($U = \frac{1}{s}$). Controller parameters were selected by tuning rules.

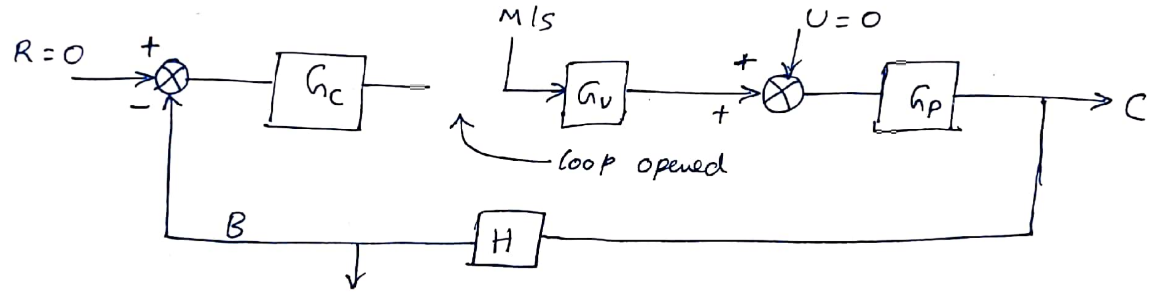




Cohen and Coon Rules (C-C)

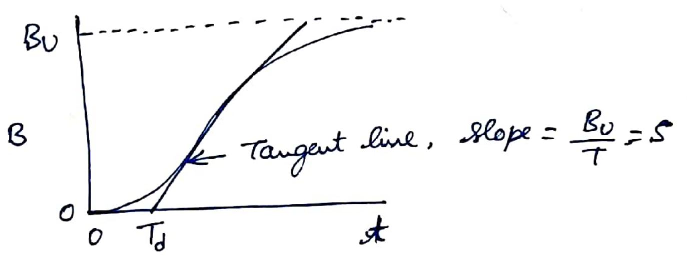
usually applied for open-loop, in which the control action is removed from the controller by placing it in manual mode and an open-loop transient is induced by a step change in the signal to the valve.

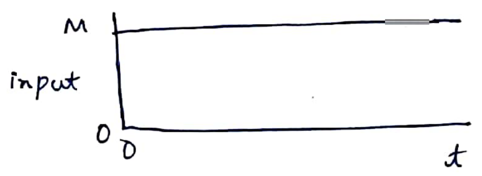
Fig. shows a typical control loop in which the control action is removed and the loop opened for the purpose of introducing a step change (M/S) to the valve. The step response is recorded at the output of the measuring element.



The step change to the valve is conveniently provided by the output from the controller, which is in manual mode.

The response of the system (including the valve, process & measuring element) is called the process reaction curve; a typical process reaction curve exhibits an S-shape as in fig.





The C-C method is summarized as follows:

1. After the process reaches S.S. at the normal level of operation, switch the controller to manual. In a modern controller, the controller output will remain at the same value after switching as it had before switching. (This is called "bumpless" transfer.)

2. With the controller in manual, introduce a small step change in the controller output that goes to the value and record the transient, which is the process reaction curve.

3. Draw a straight line tangent to the curve at the point of inflection, as in fig. The intersection of the tangent line with the time axis is the apparent transport lag (T_d); the apparent first-order time constant (T) is obtained from.

$$T = B_u / S.$$

where B_u is the ultimate value of B at large t and S is the slope of the tangent line. The steady-state gain that relates B to M in 1st fig. is given by.

$$K_p = B_u / M$$

4. Using the values of K_p, T, and T_d from step 3, the controller settings are found from the relations given in table.

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Cohen - Coon Controller Settings

Type of Control	Parameter Setting
Proportional (P)	$K_c = \frac{1}{K_p}\frac{T}{T_d}\left(1 + \frac{T_d}{3T}\right)$
Proportional-integral (PI)	$K_c = \frac{1}{K_p}\frac{T}{T_d}\left(\frac{9}{10} + \frac{T_d}{12T}\right)$
	$\tau_I = T_d \frac{30+3T_d/T}{9+20T_d/T}$
Proportional-derivative (PD)	$K_c = \frac{1}{K_p}\frac{T}{T_d}\left(\frac{5}{4} + \frac{T_d}{6T}\right)$
	$\tau_D = T_d \frac{6-2T_d/T}{22+3T_d/T}$
Proportional-integral-derivative (PID)	$K_c = \frac{1}{K_p}\frac{T}{T_d}\left(\frac{4}{3} + \frac{T_d}{4T}\right)$
	$\tau_I = T_d \frac{32+6T_d/T}{13+8T_d/T}$
	$\tau_D = T_d \frac{4}{11+2T_d/T}$

The rationale for the C-C tuning method begins with the representation of the S-shaped process reaction curve by a first-order with transport lag model; thus,

$$G_p(s) = \frac{K_p e^{-T_d s}}{Ts+1}$$

using the system expressed by this eqn, Cohen & Coon obtained by theoretical means the controller settings given by above table. Their computations required that the response have $\frac{1}{4}$ decay ratio, minimum offset, minimum area under the load response curve, & other favorable properties.

Proportional Control: Produces an overshoot followed by an oscillatory response, which levels out at a value that does not equal the set point; this ultimate displacement from the set point is the offset.

Proportional - Derivative Control: Response exhibits a smaller overshoot and a smaller period of oscillation compared to the response for Proportional Control. The offset that still remains is less than that for proportional control.

Proportional - Integral Control: The response has about the same overshoot as proportional control, but the period is larger; however, the response returns to the set point (offset = 0) after a relatively long settling time. The most beneficial influence of the integral action in the controller is the elimination of offset.

Proportional - Integral - Derivative Control: The response has lower overshoot and returns to the set point more quickly than the responses for the other types of controllers.

Integral action present in PI & PID Controllers eliminates offset. The addition of derivative action speeds up the response by contributing to the controller output a component of the signal that is proportional to the rate of change of the process variable.

For simple, low order (I & II) processes that can tolerate some offset, P & PD control is satisfactory. For processes that cannot tolerate offset & are of low order, PI control is required. For processes that are of high order ($\bar{c}$ transport lag or many first-order lags in series), PID control is needed to prevent large overshoot & long settling time.

Earlier, usually the controllers were based on price.

Pneumatic & electronic controllers $\bar{c}$ proportional action → least cost.

& $\bar{c}$ PID action → most expensive.

Today price is not mattered in the industries. All modern microprocessor based controller comes with three actions, as well as other functions such as lead-lag and transport lag.

A response that gives min. overshoot, $\frac{1}{4}$ decay ratio is often considered ~~for~~ as a satisfactory response.

Criteria to evaluate a response of a control system

1) Integral of the square of the error w.r.t. time (ISE)

$$ISE = \int_0^\infty e^2 dt.$$

e → error.

2) Integral of the absolute value of error (IAE)

$$IAE = \int_0^\infty |e| dt.$$

3) Integral of time-weighted absolute error (ITAE)

$$ITAE = \int_0^\infty |e| t \, dt.$$

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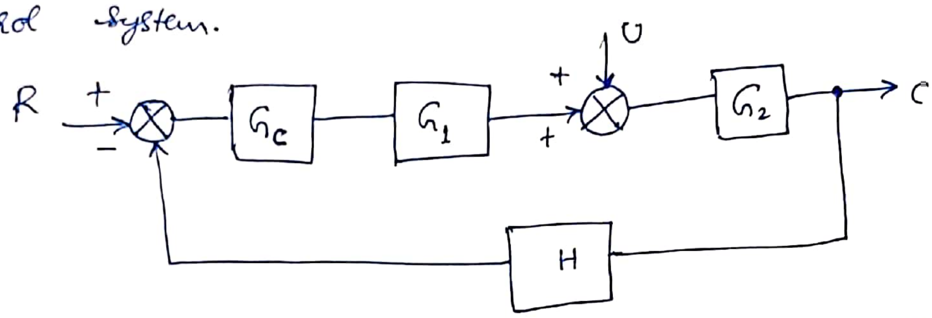
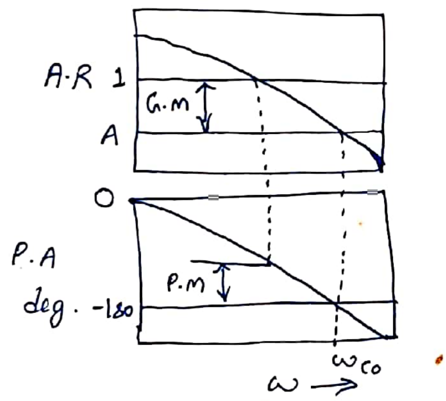
Ziegler - Nichols Controller Settings.

Consider selection of a Controller G_c for the general Control system.



First plot the Bode diagram for the final Control element, the process, and the measuring element in series, $G_1 G_2 H(j\omega)$. The Controller is omitted from this plot. If the diagram appears as below.



As in the figure, at the crossover frequency, the overall gain is A. According to the Bode Criterion, then, the gain of a proportional Controller which would Cause the system of above fig. to be on the verge of instability is $1/A$.

This quantity is defined to be the ultimate gain K_U.

Thus, $K_U = \frac{1}{A}$.

The ultimate Period P_U is defined as the Period of the sustained cycling that would occur if a proportional Controller with gain K_U were used.

$$P_U = \frac{2\pi}{\omega_{co}} \quad \text{time/cycle}$$

The factor of 2π appears, so that P_U will be in units of time per cycle rather than time per radian.

K_U & P_U are easily determined from the Bode diagram.

The Ziegler - Nichols settings for Controllers are determined directly from K_U & P_U according to the rules in table.

Ziegler–Nichols Controller Settings

Type of Control	$G_c(s)$	K_c	τ_I	τ_D
Proportional	K_c	$0.5 K_u$		
Proportional-Integral (PI)	$K_c\left(1+\frac{1}{\tau_I s}\right)$	$0.45 K_u$	$\frac{P_u}{1.2}$	
Proportional-Integral-Derivative (PID)	$K_c\left(1+\frac{1}{\tau_I s}+\tau_D s\right)$	$0.6 K_u$	$\frac{P_u}{2}$	$\frac{P_u}{8}$

Unfortunately, specifications of K_c & τ_D for PD control cannot be made using only K_u & P_u.

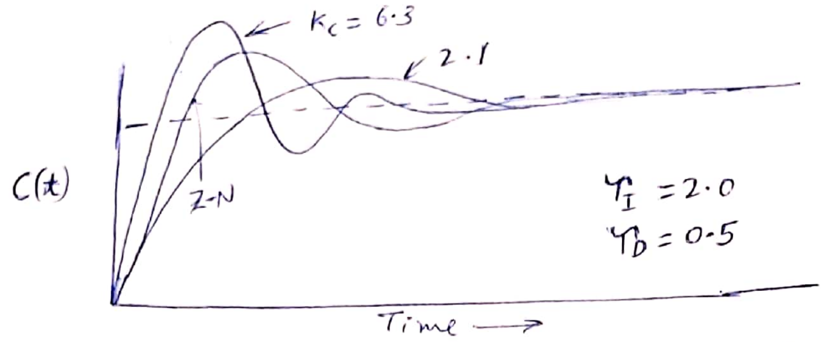
In general, the values $0.6 K_u$ and $\frac{P_u}{8}$, which correspond to the limiting case of no integral action in a three-mode controller, are too conservative. That is, the resulting system will be too stable.

Other methods for PD controller are also difficult. One problem is the selection of τ_D for maximum K_c at $30°$ phase margin.

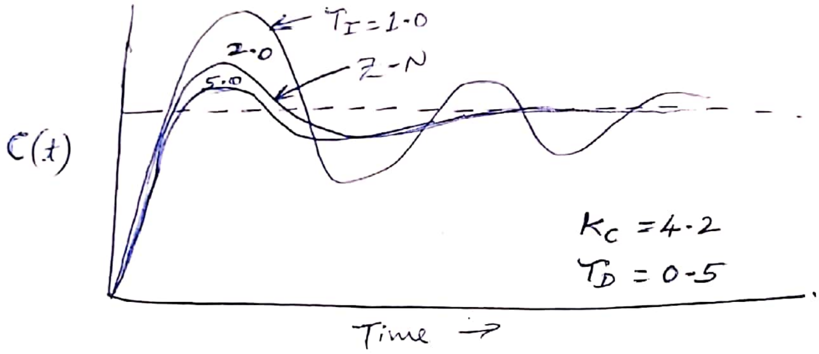
The reasoning behind the Ziegler–Nichols selection of values of K_c is relatively clear. In the case of proportional control only, a gain margin of 2 is established. The addition of integral action introduces more phase lag at all frequencies; hence a lower value of K_c is required to maintain roughly the same gain margin.

Adding derivative action introduces phase lead. Hence, more gain may be tolerated.

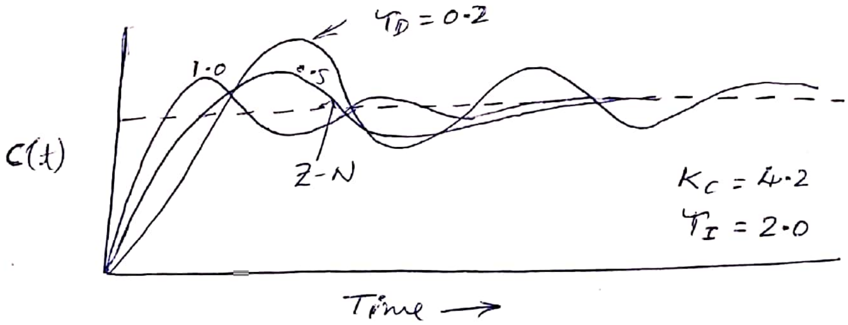
However, by and large the Ziegler–Nichols settings are based on experience with typical processes and should be regarded as first estimates.



It shows that overshoot may be reduced by decreasing K_c at the expense of a considerably more sluggish response.



The overshoot may be reduced by increasing τ_I (decreasing integral action) at a lesser expense in speed of response. Thus, for $\tau_I = 5$ min, the overshoot is reduced to 20% without a serious sacrifice in speed.

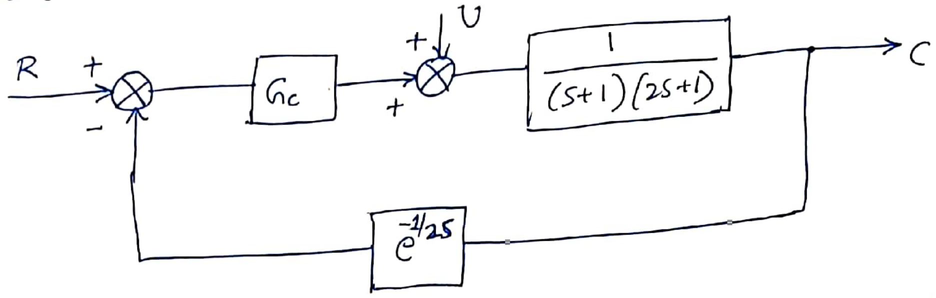


The overshoot cannot be significantly reduced by changing τ_D as in fig. However, the speed of response may be significantly increased by increasing the derivative action, at the expense of more oscillation before the response has settled (higher decay ratio, lower period)

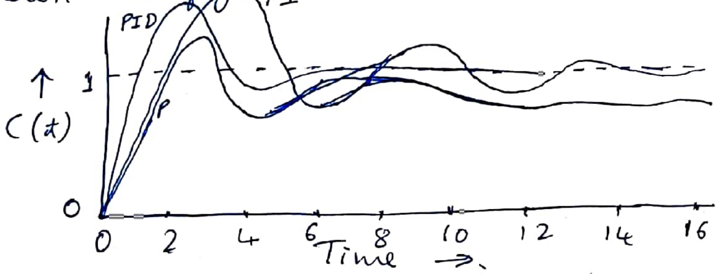
From all figures, it may be concluded that, to decrease overshoot without seriously slowing the response, a combination of changes should be made. A possible combination, which should be tried, is to reduce K_c slightly and to increase τ_I & τ_D moderately. These changes would probably be tried on the actual reactor system when it is put into operation.

Transient Responses

Consider the two-tank reactor system.



Responses of $C(t)$ to a unit-step change in $R(t)$ are shown in fig



Values of the various parameters determined for the responses of above fig are as in table below.

Control	Overshoot	Decay ratio	Rise time, min	Response time, min	Period of oscillation, min	Offset.
P	0.49	0.26	1.3	10.4	5.0	0.21
PI	0.46	0.29	1.5	11.8	5.5	0
PID	0.42	0.05	0.9	4.9	5.0	0

The figures are transient responses to step changes in set point for the three-mode PID control. They show the effects of individually varying the three control parameters K_c, τ_I, τ_D.

Fig is presented for two purposes:

i) To illustrate that the Ziegler-Nichols controller settings should be regarded as first guesses rather than fixed values &

ii) To show the effects of changing the various controller settings.

Tuning Rules

Ziegler - Nichols Rules (Z-N)

closed - loop tuning method, Rules for actual application to a real process.

1. After the process reaches steady state at the normal level of operation, remove the integral and derivative modes of controller, leaving only proportional control.

   For some PID controllers. τ_I is set to its max. value & τ_D to min. value.

   For modern controllers (microprocessor-based), the integral & derivative modes can be removed completely from the controller.

2. Select a value of proportional gain (K_c), disturb the system, and observe the transient response. If the response decays, select a higher value of K_c and again observe the response of the system. Continue increasing the gain in small steps until the response first exhibits a sustained oscillation.

   The value of gain and the period of oscillation that correspond to the sustained oscillation are the ultimate gain (K_{cu}) and the ultimate period (P_u)

3. From the values of K_{cu} and P_u found in previous step, use the Ziegler - Nichols rules given in Table. to determine controller setting (K_c, τ_I, τ_D)

   Z - N rules generally provide conservative (& safe) controller settings. Z - N settings should be considered as only approximate settings for satisfactory control. Fine tuning of the controller settings is usually required to get an improved control response.

Z-N. C. S	$G_c(s)$	K_c	τ_I	τ_D
P	K_c	$0.5 K_u$		
PI	$K_c(1 + \frac{1}{\tau_I s})$	$0.45 K_u$	$\frac{P_u}{1.2}$	
PID	$K_c(1 + \frac{1}{\tau_I s} + \tau_D s)$	$0.6 K_u$	$\frac{P_u}{2}$	$\frac{P_u}{8}$

Precautions to take in applying the Z-N method.

Precautions concerned with type & size of disturbance in set pt.

Simple way to introduce a disturbance is to move the set point away from the control variable for a short time and then return the set pt. to its original value.

An alternate type of disturbance would be to introduce a small step change in set point. If step changes in set point are used to induce transients, the successive step changes should alternate around the normal operating point of the process. It is also important to make the disturbance as small as possible, especially as the gain of the controller is increased, so that the value & other components do not exceed their physical limits.

When the valve moves to its limits during a closed-loop transient, we say that the valve saturates. Under these conditions, a sustained oscillation occurs, which is called a limit cycle.

If a limit cycle occurs, the gain that produces it and the period of the cycle should not be used in the Ziegler-Nichols rules.

Since the limit cycle will appear ~~to~~ the observer to be the same as a sustained oscillation when the system is on the verge of instability, the inexperienced person will often mistakenly use the information derived from the limit cycle (controller gain & period) to obtain controller settings. A simple way to know if one has a limit cycle is to observe the swing in pressure to the valve. If the limits of the valve (eg. 3 Psi to 15 Psi) are reached repeatedly during the oscillatory response, one has a limit cycle & the controller gain and period should not be used to det^n controller settings. Hence step 2 states K_c should be increased in

small steps until the response first exhibits a sustained oscillation.